

# A New Method of Matrix Spectral Factorization <sup>1</sup>

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**Abstract**—A new method of matrix spectral factorization is proposed which reliably computes an approximate spectral factor of any matrix spectral density that admits spectral factorization.

**Index Terms**—Matrix spectral factorization algorithm.

## 1. INTRODUCTION

Spectral factorization plays a prominent role in a wide range of fields in Communications, System Theory, Control Engineering and so on. In the scalar case arising for single input and single output systems, the factorization problem is relatively easy and several classical algorithms exist to tackle it (see the survey paper [17]) together with reliable information on their software implementations [8]. There are also some recent claims as to their improvement [2]. Matrix spectral factorization which arises for multi-dimensional systems is essentially more difficult (see Sect. 2, where the mathematical reasons of this fact are explained). Since Wiener's original efforts [19] to create a sound computational method of such factorization, tens of different algorithms have appeared in the literature (see the survey papers [16], [17] and the references therein), but none of them is thought to have an essential superiority over all others (see [16, p. 1077], [14, p. 206]). Besides, most of these algorithms impose extra restrictions on matrix spectral densities (e.g., to be real or rational or nonsingular on the boundary), while the Paley-Wiener necessary and sufficient condition (see (2)) will do for the existence of spectral factorization (see Sect. 2).

In the present paper, a new computational method of matrix spectral factorization is developed. The proposed algorithm can be applied to any matrix spectral density satisfying the Paley-Wiener condition. It should be said that the branch of mathematics where the spectral factorization problem is posed in its general non-rational setting (see Sect. 2) is the theory of Hardy spaces (see Sect. 3), and this method is completely worked out in the framework of Hardy spaces, which added to its effectiveness.

To describe our method of  $r \times r$  matrix spectral factorization in a few words, it carries out spectral factorization of  $m \times m$  left-upper submatrices step-by-step,  $m = 1, 2, \dots, r$ . It is shown that in this process the decisive role is played by unitary matrix functions of certain structure (see Theorem 1), which removes many technical difficulties connected with computation. The explicit construction of such matrices in Theorem 2 is an essential component of the algorithm. A close relationship of these unitary matrix functions with compactly supported wavelets has recently been

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<sup>1</sup>This paper includes the detailed proofs for an innovative method for matrix spectral factorization that can be used in numerous applications, including Filtering, Data Compression, and Wireless Communications. A U.S. patent application has been submitted for this innovation through the Technology Commercialization Center of the University of Maryland.

discovered, which makes it possible to construct compact wavelets in a fast and reliable way and to completely parameterize them (see [6]).

Preliminary numerical simulations confirm the potential of the proposed algorithm (see Sect. 7).

The algorithm was announced in [3] and, for second order matrices, described in [12].

## 2. FORMULATION OF THE PROBLEM

A series of papers [18], [19], [10], [11] led to the following

**Wiener Matrix Spectral Factorization Theorem:** Let

$$(1) \quad S(t) = \begin{pmatrix} s_{11}(t) & s_{12}(t) & \cdots & s_{1r}(t) \\ s_{21}(t) & s_{22}(t) & \cdots & s_{2r}(t) \\ \vdots & \vdots & \ddots & \vdots \\ s_{r1}(t) & s_{r2}(t) & \cdots & s_{rr}(t) \end{pmatrix},$$

$t \in \mathbb{T}$ , be a positive definite (a.e.) integrable matrix function,  $0 < S(t) \in L_1(\mathbb{T})$ , which satisfies the condition

$$(2) \quad \log \det S(t) \in L_1(\mathbb{T}).$$

Then it admits a *spectral factorization*

$$(3) \quad S(t) = S^+(t)S^-(t) = S^+(t)(S^+(t))^*,$$

where  $S^+$  is an  $r \times r$  outer analytic matrix function from the Hardy space  $H_2$  and  $S^-(z) = (S^+(1/\bar{z}))^*$ ,  $|z| > 1$ . It is assumed that (3) holds a.e. on  $\mathbb{T}$ . (The factorization (3) is called *left* since the analytic inside  $\mathbb{T}$  factor stands on the left-hand side. The *right* spectral factorization of  $S$  can be obtained by the left factorization of  $S^T$ .)

The sufficient condition (2) is also a necessary one for the factorization (3) to exist (see Sect. 3).

A spectral factor  $S^+(z)$  is unique up to a constant right unitary multiplier (see, e.g., [5]), and the unique spectral factor with an additional requirement that  $S^+(0)$  be positive definite is called *canonical*.

After the proof of the existence of matrix spectral factorization, the computation of the spectral factor for a given matrix spectral density has become a challenging problem due to its applications in practice.

In the scalar case,  $r = 1$ , the canonical spectral factor  $S^+ \in H_2$  can be explicitly written by the formula (see, e.g., [20; VII, 7.33])

$$(4) \quad S^+(z) = \exp \left( \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log S(e^{i\theta}) d\theta \right)$$

and it is relatively easy to compute  $S^+$  approximately. However, there is no analog of this formula in the matrix case because, generally speaking,  $e^{A+B} \neq e^A e^B$  for non-commutative matrices  $A$  and  $B$ . This is the main reason for which the approximate computation of the spectral factor  $S^+$  in (3) for the matrix spectral density (1) is essentially more difficult. The present paper provides an algorithm for such computation.

The proposed method does not contribute to the improvement of (numerical) scalar spectral factorization, but employs it to fulfill matrix spectral factorization.

### 3. NOTATION AND CONVENTIONS

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ , and  $\mathbb{T} = \partial\mathbb{D}$  be the unit circle. As usual,  $L_p = L_p(\mathbb{T})$ ,  $0 < p \leq \infty$ , denotes the Lebesgue space of  $p$ -integrable complex functions defined on  $\mathbb{T}$ .  $H_p = H_p(\mathbb{D})$ ,  $0 < p \leq \infty$ , is the Hardy space of analytic functions in  $\mathbb{D}$ ,

$$H_p = \left\{ f \in \mathcal{A}(\mathbb{D}) : \sup_{r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty \right\}$$

( $H_\infty$  is the space of bounded analytic functions), and  $L_p^+ = L_p^+(\mathbb{T})$  denotes the class of their boundary functions. (All the relations for functions from  $L_p^+(\mathbb{T})$  or  $L_p(\mathbb{T})$  are assumed to hold almost everywhere.) Since there is a one-to-one correspondence between  $H_p(\mathbb{D})$  and  $L_p^+(\mathbb{T})$ ,  $p > 0$  (see, e.g., [20; VII, 7.25]), we naturally regard these two classes as identical, and thus we can speak about the values of  $f \in L_p^+(\mathbb{T})$  inside the unit circle. Furthermore, we always use the argument  $t$  for the functions defined on  $\mathbb{T}$  and the argument  $z$  for the functions defined in  $\mathbb{D}$ , so that the boundary function of  $f = f(z) \in H_p(\mathbb{D})$  is denoted by  $f = f(t) \in L_p^+(\mathbb{T})$  and we write  $f(z)|_{z=t} = f(t)$  when we wish to point out this fact. If we write only  $f$ , its domain will be clear from the context.

We have  $\log |f(t)| \in L_1(\mathbb{T})$  for each  $0 \neq f \in H_p$ ,  $p > 0$  (see, e.g., [20; VII, 7.25]), which readily implies the necessity of the condition (2) for the factorization (3) to exist since  $L_1(\mathbb{T}) \ni \log |\det S^+(t)| = \frac{1}{2} \log \det S(t)$ .

The  $n$ th Fourier coefficient of an integrable function  $f \in L_1(\mathbb{T})$  is denoted by  $c_n(f)$ . For  $p \geq 1$ ,  $L_p^+(\mathbb{T})$  coincides with the class of functions from  $L_p(\mathbb{T})$  whose Fourier coefficients with negative indices are equal to zero. We also deal with  $L_p^-(\mathbb{T}) = \{\bar{f} : f \in L_p^+(\mathbb{T})\} = \{f \in L_p(\mathbb{T}) : c_n(f) = 0 \text{ whenever } n > 0\}$ . The set of trigonometric polynomials is denoted by  $\mathcal{P}$ , i.e.  $f \in \mathcal{P}$  if  $f$  has only a finite number of nonzero Fourier coefficients. Also let  $\mathcal{P}^\pm := \mathcal{P} \cap L_\infty^\pm$ ,  $\mathcal{P}_N := \{f \in \mathcal{P} : c_n(f) = 0 \text{ whenever } |n| > N\}$ , and  $\mathcal{P}_N^\pm = \mathcal{P}_N \cap \mathcal{P}^\pm$ . Obviously,  $f \in \mathcal{P}_N^+ \Leftrightarrow \bar{f} \in \mathcal{P}_N^-$ .

For  $f(t) = \sum_{n=-\infty}^{\infty} c_n t^n \in L_2(\mathbb{T})$ , let  $P^+ f(t)$ ,  $P^- f(t)$ , and  $P_N f(t)$  be the projections  $\sum_{n=0}^{\infty} c_n t^n$ ,  $\sum_{n=-\infty}^0 c_n t^n$ , and  $\sum_{n=-N}^N c_n t^n$ , respectively, on  $L_2^+(\mathbb{T})$ ,  $L_2^-(\mathbb{T})$ , and  $\mathcal{P}_N$ .

The superscript "+" (resp. "-") of a function  $f^+$  (resp.  $f^-$ ) emphasizes that this function belongs to  $L_p^+$  (resp.  $L_p^-$ ).

The norms  $\|\cdot\|_{L_p}$  and  $\|\cdot\|_{H_p}$  are defined in a usual way.

If  $M$  is a matrix, then  $\overline{M}$  denotes the matrix with conjugate entries and  $M^* := \overline{M}^T$ . If  $M$  is positive definite,  $M > 0$ , then the unique  $M_0 > 0$  that satisfies  $M_0 M_0^* = M$  is denoted by  $\sqrt{M}$ .

If  $M$  is an  $r \times r$  matrix and  $m \leq r$ , then  $(M)_{m \times m}$  is assumed to be the  $m \times m$  upper-left submatrix of  $M$ .

An  $r \times r$  matrix  $U$  is called unitary if  $UU^* = U^*U = I_r$ , where  $I_r$  stands for the  $r$ -dimensional unit matrix. Obviously the entries of a unitary matrix are bounded by 1.

A matrix function  $M(t)$  defined on  $\mathbb{T}$  is called positive definite or unitary if it is such for almost all  $t \in \mathbb{T}$ .  $M(t)$  is said to belong to some class, say,  $L_p^+(\mathbb{T})$  (we write  $M(t) \in L_p^+(\mathbb{T})$ ) if its entries belong to this class.  $P_N M(t)$  denotes the matrix function whose entries are the projections of the entries of  $M(t)$  on  $\mathcal{P}_N$ . A sequence of matrix functions is said to be convergent in  $L_p$ -norm if their entries are convergent in this norm.

The class of  $r \times r$  unitary matrix functions  $U(t)$ ,

$$(5) \quad U(t)U^*(t) = I_r \text{ a.e.},$$

is denoted by  $\mathcal{U}_r(\mathbb{T})$ , and  $\mathcal{SU}_r(\mathbb{T})$  stands for the subclass of those  $U(t) \in \mathcal{U}_r(\mathbb{T})$  the determinants of which are equal to 1,

$$(6) \quad \det U(t) = 1 \text{ a.e.}$$

The set of outer analytic functions from the Hardy space  $H_p$ ,  $p > 0$ , is denoted by  $\mathcal{O}_p$ . Recall that  $f \in \mathcal{O}_p$  if and only if  $0 \notin f \in H_p$  and

$$f(z) = c \cdot \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |f(e^{i\theta})| d\theta \right), \quad |c| = 1.$$

(From this definition and Hölder's inequality it follows that if  $f \in \mathcal{O}_p$  and  $g \in \mathcal{O}_q$ , then  $fg \in \mathcal{O}_{(p+q)/pq}$ .) Clearly,  $f(z) \neq 0$  for each  $z \in \mathbb{D}$  and  $|f(t)| > 0$  for a.a.  $t \in \mathbb{T}$  if  $f \in \mathcal{O}_p$ . The set of functions  $f \in \mathcal{O}_p$  which are positive at the origin (which happens when  $c = 1$  in the above definition) is denoted by  $\mathcal{O}_p^0$ . We say that a  $r \times r$  matrix function  $M(t) \in H_p$ ,  $p \geq 0$ , is outer if its determinant belongs to  $\mathcal{O}_{p/r}$ . This definition coincides with some other equivalent definitions of outer matrix functions (see, e.g., [11]).  $M(t) \in \mathcal{O}_{p/r}^0$  means that  $M(0) > 0$  in addition.

$f_n \rightrightarrows f$  means that  $f_n$  converges to  $f$  in measure.

$\langle \cdot, \cdot \rangle_m$  and  $\|\cdot\|_{\mathbb{C}^m}$  denote the usual scalar product and the norm, respectively, in the  $m$ -dimensional complex space  $\mathbb{C}^m$ .

$\delta_{ij}$  stands for the Kronecker delta.

To conclude the section, we formulate a simple statement from the Lebesgue integral theory in the best suitable form for further references.

STATEMENT 1. *Let  $f_n(t) \in L_2(\mathbb{T})$ ,  $n = 1, 2, \dots$ ,  $\|f_n(t) - f(t)\|_{L_2} \rightarrow 0$ ,  $u_n(t) \in L_\infty(\mathbb{T})$ ,  $u_n(t) \leq 1$ ,  $n = 1, 2, \dots$ , and  $u_n(t) \rightrightarrows u(t)$ . Then  $\|f_n(t)u_n(t) - f(t)u(t)\|_{L_2} \rightarrow 0$  (see, e.g., [9; §26, Th. 3]).*

#### 4. MATHEMATICAL BACKGROUND OF THE METHOD

In this section we formulate some statements needed to describe our method. Most of the proofs are given in the next sections.

The uniqueness of spectral factorization (3) mentioned in Sect. 2 means that  $S^+(z) \cdot U$  is also a spectral factor for any (constant) unitary matrix  $U$ , and if  $S_1^+(z)$  and  $S_2^+(z)$  are two spectral factors, then  $S_1^+(z) = S_2^+(z)U$  for some unitary matrix  $U$  (see, e.g., [5]). Since for any  $r \times r$  non-singular matrix  $S$  there exists a unique unitary matrix  $U$  which makes the product  $SU$  positive definite (see, e.g. [7; IX §14]), the canonical

spectral factor  $S_c^+(z)$  (with an additional requirement that  $S_c^+(0)$  be positive definite) is unique. Namely,

$$(7) \quad S_c^+(z) = S^+(z)(S^+(0))^{-1}\sqrt{S^+(0)(S^+(0))^*}$$

for any spectral factor  $S^+(z)$ . (Other uniqueness restrictions on  $S^+$  can be imposed so that  $S^+(0)$  would be, for example, lower triangular with positive entries on the diagonal.) The following lemma can be applied for the approximation of the canonical spectral factor after the approximate computation of an arbitrary spectral factor.

**Lemma 1.** *Let  $S^+(t)$  be a spectral factor of (1) and let  $S_n^+(t) \in H_2$ ,  $n = 1, 2, \dots$ , be such that*

$$(8) \quad \|S_n^+(t) - S^+(t)\|_{H_2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*Then*

$$(9) \quad \|S_n^+(z)(S_n^+(0))^{-1}\sqrt{S_n^+(0)(S_n^+(0))^*} - S_c^+(z)\|_{H_2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*Proof.* Since  $S^+(0)$  is non-singular and (8) implies that  $S_n^+(0) \rightarrow S^+(0)$ , we have  $\sqrt{S_n^+(0)(S_n^+(0))^*} \rightarrow \sqrt{S^+(0)(S^+(0))^*}$  and  $(S_n^+(0))^{-1} \rightarrow (S^+(0))^{-1}$ . Therefore (9) follows from (8) and (7).  $\square$

The following lemma is used several times throughout the paper.

**Lemma 2.** *Let  $M(t)$  be any  $m \times m$  matrix function from  $L_2(\mathbb{T})$  satisfying*

$$(10) \quad \det M(t) \in \mathcal{O}_{2/m} \subset H_{2/m} = L_{2/m}^+(\mathbb{T}).$$

*If  $U(t) \in \mathcal{SU}_m(\mathbb{T})$  is such that*

$$(11) \quad M(t)U(t) \in L_2^+(\mathbb{T})$$

*holds, then  $M(t)U(t)$  is a spectral factor of  $M(t)M^*(t)$ .*

*Proof.* Taking into account (5), we have

$$M(t)U(t) \cdot (M(t)U(t))^* = M(t)U(t)U^*(t)M^*(t) = M(t)M^*(t).$$

In view of (11),  $MU$  can be extended inside  $\mathbb{T}$ . Hence, by virtue of (6),

$$\det(MU)(z)|_{z=t} = \det(M(t)U(t)) = \det M(t) = \det M(z)|_{z=t}$$

Thus  $\det(MU)(z)$  is an outer analytic function (see (10)) and lemma holds.  $\square$

This proof gives rise to

**COROLLARY 1.** *Let  $M(t)$  be any  $m \times m$  matrix function from  $L_2(\mathbb{T})$  satisfying  $\det M(t) \in L_{2/m}^+$ . If  $U(t) \in \mathcal{SU}_m(\mathbb{T})$  is such that (11) holds, then*

$$(12) \quad \det(MU)(z) = (\det M)(z), \quad |z| < 1.$$

It should be pointed out that on the left-hand side of (12) we first extend  $MU$  inside  $\mathbb{T}$  and then compute its determinant, while on the right-hand side we first take the determinant of  $M(t)$  and then extend it inside  $\mathbb{T}$ .

The following two theorems play a decisive role in our method.

**Theorem 1.** For every  $m \times m$  matrix function  $F(t) \in L_2(\mathbb{T})$  of the form

$$(13) \quad F(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1(t) & \zeta_2(t) & \zeta_3(t) & \cdots & \zeta_{m-1}(t) & f^+(t) \end{pmatrix},$$

where

$$(14) \quad \zeta_j(t) \in L_2(\mathbb{T}), \quad j = 1, 2, \dots, m-1, \quad \text{and} \quad f^+(t) \in \mathcal{O}_2^0 \subset L_2^+(\mathbb{T}),$$

there exists a unique  $U_F(t) \in \mathcal{SU}_m(\mathbb{T})$  of the form

$$(15) \quad U_F(t) = \begin{pmatrix} u_{11}^+(t) & u_{12}^+(t) & \cdots & u_{1,m-1}^+(t) & u_{1m}^+(t) \\ u_{21}^+(t) & u_{22}^+(t) & \cdots & u_{2,m-1}^+(t) & u_{2m}^+(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}^+(t) & u_{m-1,2}^+(t) & \cdots & u_{m-1,m-1}^+(t) & u_{m-1,m}^+(t) \\ \overline{u_{m1}^+(t)} & \overline{u_{m2}^+(t)} & \cdots & \overline{u_{m,m-1}^+(t)} & \overline{u_{mm}^+(t)} \end{pmatrix},$$

$$(16) \quad u_{ij}^+(t) \in L_\infty^+(\mathbb{T}), \quad i, j = 1, 2, \dots, m,$$

such that

$$(17) \quad F(t)U_F(t) = F_c^+(t) \in \mathcal{O}_2^0 \subset L_2^+(\mathbb{T}),$$

where  $F_c^+(t)$  is the canonical spectral factor of  $F(t)F^*(t)$ .

The proof of Theorem 1 relying on the existence of spectral factorization is relatively easy (see [3]). The core of the proposed matrix spectral factorization method is the constructive proof of Theorem 1 based on the following idea. We approximate  $F(t)$  in  $L_2$  cutting off the tails of Fourier expansions of the functions  $\zeta_j(t)$ ,  $j = 1, 2, \dots, m-1$ , and  $f^+(t)$ . Namely, for a matrix function of the form (13), (14), let  $F^{(N)}(t)$  be  $P_N F(t)$ , i.e.

$$(18) \quad F^{(N)}(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1^{(N)}(t) & \zeta_2^{(N)}(t) & \zeta_3^{(N)}(t) & \cdots & \zeta_{m-1}^{(N)}(t) & f_{(N)}^+(t) \end{pmatrix},$$

where

$$f_{(N)}^+(t) = \sum_{n=0}^N c_n(f^+)t^n, \quad \text{and} \quad \zeta_j^{(N)}(t) = \sum_{n=-N}^N c_n(\zeta_j)t^n,$$

$j = 1, 2, \dots, m-1$ . It is obvious that,

$$(19) \quad \|\zeta_j^{(N)}(t) - \zeta_j(t)\|_{L_2} \rightarrow 0, \quad \|f_{(N)}^+(t) - f^+(t)\|_{L_2} \rightarrow 0$$

or, equivalently,

$$(20) \quad \|F^{(N)}(t) - F(t)\|_{L_2} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

We will multiply (18) by the polynomial unitary matrix function which eliminates the Fourier coefficients with negative indices of the product. Furthermore, we prove the following theorem for matrix-functions  $F^{(N)}(t)$ ,  $N = 1, 2, \dots$ , which involves the limiting case too.

**Theorem 2. (a)** *Let  $N$  be any positive integer, and let a matrix function  $F^{(N)}(t) \in \mathcal{P}_N$  of the form (18) be such that*

$$(21) \quad \zeta_j^{(N)}(t) \in \mathcal{P}_N, \quad j = 1, 2, \dots, m-1, \text{ and } f_{(N)}^+(t) \in \mathcal{P}_N^+, \quad f_{(N)}^+(0) > 0.$$

*Then there exists and one can explicitly construct*

$$(22) \quad U_{F^{(N)}}(t) \in \mathcal{SU}_m(\mathbb{T})$$

*of the form (15) such that*

$$(23) \quad u_{ij}^+(t) \in \mathcal{P}_N^+, \quad i, j = 1, 2, \dots, m,$$

$$(24) \quad F^{(N)}(t)U_{F^{(N)}}(t) \in \mathcal{P}^+,$$

*and*

$$(25) \quad F^{(N)}U_{F^{(N)}}(0) > 0.$$

**(b)** *Given an arbitrary sequence of matrix functions  $F^{(N)}(t)$ ,  $N = 1, 2, \dots$ , of the form (18), (21) which converges in  $L_2$  to  $F(t)$  (i.e. (20) holds) of the form (13), (14), we have*

$$(26) \quad \|F^{(N)}(t)U_{F^{(N)}}(t) - F_c^+(t)\|_{H_2} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

*Furthermore, the sequence  $U_{F^{(N)}}(t)$ ,  $N = 1, 2, \dots$ , is convergent in measure. The limiting matrix function  $U(t) \in \mathcal{SU}_m(\mathbb{T})$  satisfies the conditions imposed on  $U_F(t)$  in Theorem 1, and therefore  $U(t)$  coincides with  $U_F(t)$ . Consequently, we have*

$$(27) \quad U_{F^{(N)}}(t) \rightrightarrows U_F(t).$$

The constructive proof of Theorem 2 (a) given in Sect 5, which computes explicitly and in a fast reliable way the coefficients of the functions  $u_{ij}^+(t) \in \mathcal{P}_N^+$  in (23), is the essence of the proposed algorithm. The part (b) of the theorem involves the algorithm convergence properties and is proved in Appendix A. We point out the fact that Theorem 2 (b) includes also the proof of Theorem 1.

## 5. A CONSTRUCTIVE PROOF OF THEOREM 2 (a)

Throughout this section it is assumed that  $N$  is fixed and  $\zeta_j(t) := \zeta_j^{(N)}(t)$ ,  $f^+(t) := f_{(N)}^+(t)$ , and  $F(t) := F^{(N)}(t)$ .

For given functions  $\zeta_j(t)$ ,  $j = 1, 2, \dots, m-1$ , and  $f^+(t)$  satisfying (21), we consider the following system of  $m$  conditions, which plays a key role in the proof,

$$(28) \quad \begin{cases} \zeta_1(t)x_m^+(t) - f^+(t)\overline{x_1^+(t)} \in \mathcal{P}^+, \\ \zeta_2(t)x_m^+(t) - f^+(t)\overline{x_2^+(t)} \in \mathcal{P}^+, \\ \cdot \quad \cdot \quad \cdot \\ \zeta_{m-1}(t)x_m^+(t) - f^+(t)\overline{x_{m-1}^+(t)} \in \mathcal{P}^+, \\ \zeta_1(t)x_1^+(t) + \zeta_2(t)x_2^+(t) + \dots + \zeta_{m-1}(t)x_{m-1}^+(t) + f^+(t)\overline{x_m^+(t)} \in \mathcal{P}^+, \end{cases}$$

where the vector function  $(x_1^+(t), x_2^+(t), \dots, x_m^+(t))^T$  is unknown.

We say that a vector function

$$(29) \quad \mathbf{u}^+(t) = (u_1^+(t), u_2^+(t), \dots, u_m^+(t))^T \in \mathcal{P}_N^+$$

is a solution of (28) if and only if all the conditions in (28) are satisfied whenever  $x_i^+(t) = u_i^+(t)$ ,  $i = 1, 2, \dots, m$ . Observe that the set of solutions of (28) is a linear subspace of  $m$ -dimensional vector-valued functions defined on  $\mathbb{T}$ .

For the vector function (29), we define the modified vector function  $\widetilde{\mathbf{u}}^+(t)$  as

$$(30) \quad \widetilde{\mathbf{u}}^+(t) = (u_1^+(t), u_2^+(t), \dots, \overline{u_m^+(t)})^T.$$

It is assumed that the modification of (30) is (29).

We make essential use of the following

**Lemma 3.** *Let (21) hold and let*

$$(31) \quad \mathbf{u}^+(t) = (u_1^+(t), u_2^+(t), \dots, u_m^+(t))^T \in \mathcal{P}_N^+ \text{ and } \mathbf{v}^+(t) = (v_1^+(t), v_2^+(t), \dots, v_m^+(t))^T \in \mathcal{P}_N^+$$

*be two (possibly identical) solutions of the system (28). Then  $\langle \widetilde{\mathbf{u}}^+(t), \widetilde{\mathbf{v}}^+(t) \rangle_m$  is the same for each  $t \in \mathbb{T}$ , i.e.*

$$(32) \quad \sum_{i=1}^{m-1} u_i^+(t)\overline{v_i^+(t)} + \overline{u_m^+(t)}v_m^+(t) = \text{const}.$$

*Proof.* Substituting the functions  $v^+$  into the first  $m-1$  conditions and the functions  $u^+$  in the last condition of (28), and then multiplying the first  $m-1$  conditions by  $u^+$  and the last condition by  $v_m^+$ , we get

$$\begin{cases} \zeta_1 v_m^+ u_1^+ - f^+ \overline{v_1^+} u_1^+ \in \mathcal{P}^+, \\ \zeta_2 v_m^+ u_2^+ - f^+ \overline{v_2^+} u_2^+ \in \mathcal{P}^+, \\ \cdot \quad \cdot \quad \cdot \\ \zeta_{m-1} v_m^+ u_{m-1}^+ - f^+ \overline{v_{m-1}^+} u_{m-1}^+ \in \mathcal{P}^+, \\ \zeta_1 u_1^+ v_m^+ + \zeta_2 u_2^+ v_m^+ + \dots + \zeta_{m-1} u_{m-1}^+ v_m^+ + f^+ \overline{u_m^+} v_m^+ \in \mathcal{P}^+. \end{cases}$$

Subtracting the first  $m-1$  conditions from the last condition in the latter system, we get

$$(33) \quad f^+(t) \left( \sum_{i=1}^{m-1} u_i^+(t)\overline{v_i^+(t)} + \overline{u_m^+(t)}v_m^+(t) \right) \in \mathcal{P}^+.$$



Since the second multiplier in (33) belongs to  $\mathcal{P}_N$  (see (31)), (21) and (33) imply that

$$(34) \quad \sum_{i=1}^{m-1} u_i^+(t) \overline{v_i^+(t)} + \overline{u_m^+(t)} v_m^+(t) \in \mathcal{P}_N^+.$$

We can interchange the roles of  $u$  and  $v$  in the above discussion to get in a similar manner that

$$(35) \quad \sum_{i=1}^{m-1} v_i^+(t) \overline{u_i^+(t)} + \overline{v_m^+(t)} u_m^+(t) \in \mathcal{P}_N^+.$$

It follows from relations (34) and (35) that the function in (32) belongs to  $\mathcal{P}_N^+ \cap \mathcal{P}_N^-$ , which implies (32).  $\square$

The proof of Theorem 2 (a) proceeds as follows. We search for a nontrivial polynomial solution

$$(36) \quad \mathbf{x}(t) = (x_1^+(t), x_2^+(t), \dots, x_m^+(t))^T \in \mathcal{P}_N^+$$

of the system (28), where

$$(37) \quad x_i^+(t) = \sum_{n=0}^N a_{in} t^n, \quad i = 1, 2, \dots, m,$$

and explicitly determine the coefficients  $a_{in}$ . We will find such  $m$  linearly independent solutions of (28) (see (51) below).

Equating all the non-positive Fourier coefficients of the functions on the left-hand side of (28) to zero, except the 0th coefficient of the  $j$ th function which we equate to 1, we get the following system of algebraic equations in the block matrix form which we denote by  $\mathbb{S}_j$ :

$$(38) \quad \mathbb{S}_j := \begin{cases} \Gamma_1 \cdot X_m - D \cdot \overline{X_1} = \mathbf{0}, \\ \Gamma_2 \cdot X_m - D \cdot \overline{X_2} = \mathbf{0}, \\ \cdot \quad \cdot \quad \cdot \\ \Gamma_j \cdot X_m - D \cdot \overline{X_j} = \mathbf{1}, \\ \cdot \quad \cdot \quad \cdot \\ \Gamma_{m-1} \cdot X_m - D \cdot \overline{X_{m-1}} = \mathbf{0}, \\ \Gamma_1 \cdot X_1 + \Gamma_2 \cdot X_2 + \dots + \Gamma_{m-1} \cdot X_{m-1} + D \cdot \overline{X_m} = \mathbf{0}. \end{cases}$$

Here the following matrix notation is used:

$$(39) \quad D = \begin{pmatrix} d_0 & d_1 & d_2 & \cdots & d_{N-1} & d_N \\ 0 & d_0 & d_1 & \cdots & d_{N-2} & d_{N-1} \\ 0 & 0 & d_0 & \cdots & d_{N-3} & d_{N-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & d_0 \end{pmatrix}, \quad \Gamma_i = \begin{pmatrix} \gamma_{i0} & \gamma_{i1} & \gamma_{i2} & \cdots & \gamma_{i,N-1} & \gamma_{iN} \\ \gamma_{i1} & \gamma_{i2} & \gamma_{i3} & \cdots & \gamma_{iN} & 0 \\ \gamma_{i2} & \gamma_{i3} & \gamma_{i4} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \gamma_{iN} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where

$$f^+(z) = \sum_{n=0}^N d_n z^n \quad \text{and} \quad \zeta_i(t) = \sum_{n=-N}^N \gamma_{in} t^{-n}, \quad i = 1, 2, \dots, m-1,$$

$$(40) \quad \mathbf{0} = (0, 0, \dots, 0)^T \in \mathbb{C}^{N+1}, \quad \text{and} \quad \mathbf{1} = (1, 0, 0, \dots, 0)^T \in \mathbb{C}^{N+1}.$$

The column vectors

$$(41) \quad X_i = (a_{i0}, a_{i1}, \dots, a_{iN})^T, \quad i = 1, 2, \dots, m,$$

(see (37)) are unknowns.

REMARK 1. We recall that if  $(X_1, X_2, \dots, X_m)$  defined by (41) is a solution of the system (38), then the vector function (36) defined by (37) is a solution of the system (28).

We need to show that the system  $\mathbb{S}_j$  (see (38)) has a solution for each  $j = 1, 2, \dots, m$ .

Since  $f^+(0) > 0$  (see (21)),  $\frac{1}{f^+}$  can be represented as a power series in the neighborhood of 0

$$\frac{1}{f^+(z)} = \sum_{n=0}^{\infty} b_n z^n,$$

where  $b_0 = (f^+(0))^{-1} > 0$ , and the inverse of the matrix  $D$  is

$$(42) \quad D^{-1} = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} & b_N \\ 0 & b_0 & b_1 & \cdots & b_{N-2} & b_{N-1} \\ 0 & 0 & b_0 & \cdots & b_{N-3} & b_{N-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_0 \end{pmatrix}.$$

Determining  $X_i$ ,  $i = 1, 2, \dots, m-1$ , from the first  $m-1$  equations of (38),

$$(43) \quad X_i = \overline{D^{-1}} \overline{\Gamma_i} \overline{X_m} - \delta_{ij} \overline{D^{-1}} \mathbf{1}, \quad i = 1, 2, \dots, m-1,$$

and then substituting them into the last equation of (38), we get

$$\Gamma_1 \overline{D^{-1}} \overline{\Gamma_1} \overline{X_m} + \Gamma_2 \overline{D^{-1}} \overline{\Gamma_2} \overline{X_m} + \dots + \Gamma_{m-1} \overline{D^{-1}} \overline{\Gamma_{m-1}} \overline{X_m} + D \overline{X_m} = \Gamma_j \overline{D^{-1}} \mathbf{1}$$

(it is assumed that  $\Gamma_m = \overline{D}$ , i.e. the right-hand side is equal to  $\mathbf{1}$  when  $j = m$ ) or, equivalently,

$$(44) \quad (\Theta_1 \overline{\Theta_1} + \Theta_2 \overline{\Theta_2} + \dots + \Theta_{m-1} \overline{\Theta_{m-1}} + I_{N+1}) \overline{X_m} = D^{-1} \Gamma_j \overline{D^{-1}} \mathbf{1},$$

where

$$(45) \quad \Theta_i = D^{-1} \Gamma_i, \quad i = 1, 2, \dots, m-1.$$

For each  $j = 1, 2, \dots, m$ , (44) is a linear algebraic system of  $N+1$  equations with  $(N+1)$  unknowns.

The matrices  $\Theta_i$ ,  $i = 1, 2, \dots, m-1$ , are symmetric since their entries are (see (45), (42), and (39))

$$(46) \quad \Theta_i[k, l] = \Theta_i[l, k] = \begin{cases} 0 & \text{for } k + l > N, \\ \sum_{n=0}^{N-(k+l)} b_n \gamma_{i, k+l+n} & \text{for } k + l \leq N. \end{cases}$$

Therefore  $\Theta_i \overline{\Theta_i} = \Theta_i \Theta_i^*$ ,  $i = 1, 2, \dots, m-1$ , are non-negative definite and the coefficient matrix of the system (44)

$$(47) \quad \Delta = \Theta_1 \Theta_1^* + \Theta_2 \Theta_2^* + \dots + \Theta_{m-1} \Theta_{m-1}^* + I_{N+1}$$

(which is the same for each  $j = 1, 2, \dots, m$ ) is positive definite (with all eigenvalues larger than or equal to 1). Consequently,  $\Delta$  is nonsingular,  $\det \Delta \geq 1$ , and the system (44) has a unique solution for each  $j$ . Furthermore,  $\Delta$  has a displacement structure of rank  $m$  (see Appendix B) which reduces the computational burden for solution of the system (44) from  $O(N^3)$  to  $O(mN^2)$  (see [14; App. F]).

Finding the matrix vector  $\overline{X_m}$  from (44) and then determining  $X_1, X_2, \dots, X_{m-1}$  from (43), we get the unique solution of  $\mathbb{S}_j$ . To indicate its dependence on  $j$ , we denote the solution of  $\mathbb{S}_j$  by  $(X_1^j, X_2^j, \dots, X_{m-1}^j, X_m^j)$ ,

$$(48) \quad X_i^j := (a_{i0}^j, a_{i1}^j, \dots, a_{iN}^j)^T, \quad i = 1, 2, \dots, m,$$

so that if we construct a matrix function  $V(t)$ ,

$$(49) \quad V(t) = \begin{pmatrix} v_{11}^+(t) & v_{12}^+(t) & \cdots & v_{1,m-1}^+(t) & v_{1m}^+(t) \\ v_{21}^+(t) & v_{22}^+(t) & \cdots & v_{2,m-1}^+(t) & v_{2m}^+(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{m-1,1}^+(t) & v_{m-1,2}^+(t) & \cdots & v_{m-1,m-1}^+(t) & v_{m-1,m}^+(t) \\ \overline{v_{m1}^+(t)} & \overline{v_{m2}^+(t)} & \cdots & \overline{v_{m,m-1}^+(t)} & \overline{v_{mm}^+(t)} \end{pmatrix}$$

by letting (see (48))

$$(50) \quad v_{ij}^+(t) = \sum_{n=0}^N a_{in}^j t^n, \quad 1 \leq i, j \leq m$$

(note that (49) has the structure required in Theorem 2 **(a)**; see (15), (23)), then its modified columns  $\widetilde{V}^1(t), \widetilde{V}^2(t), \dots, \widetilde{V}^{m-1}(t)$ , and  $\widetilde{V}^m(t)$ ,

$$(51) \quad \widetilde{V}^j(t) = (v_{1j}^+(t), v_{2j}^+(t), \dots, v_{m-1,j}^+(t), v_{mj}^+(t)), \quad j = 1, 2, \dots, m,$$

are solutions of the system (28) (see Remark 1). Hence, because of the last equation in (28),

$$(52) \quad F(t)V(t) \in \mathcal{P}^+$$

and, by virtue of Lemma 3,

$$(53) \quad \langle V^i(t), V^j(t) \rangle_m = c_{ij}, \quad i, j = 1, 2, \dots, m,$$

for each  $t \in \mathbb{T}$ . Besides, we have

$$(54) \quad \det V(t) = \text{const}, \quad t \in \mathbb{T}.$$

Indeed, the inclusion

$$(55) \quad \det V(t) \in \mathcal{P}$$

is obvious (see (49) and (50)). The relation (52) implies that (see (18))

$$f^+(t) \det V(t) = \det F(t) \det V(t) \in \mathcal{P}^+.$$

Thus, it follows from (21) and (55) that

$$(56) \quad \det V(t) \in \mathcal{P}^+.$$

Next we have (see (98) below)

$$(57) \quad (F^{-1})^* = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\overline{\zeta_1/f^+} \\ 0 & 1 & \cdots & 0 & -\overline{\zeta_2/f^+} \\ 0 & 0 & \cdots & 0 & -\overline{\zeta_3/f^+} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\overline{\zeta_{m-1}/f^+} \\ 0 & 0 & \cdots & 0 & 1/\overline{f^+} \end{pmatrix}$$

and

$$(58) \quad \det (F^{-1})^*(t) = (\overline{f^+(t)})^{-1}.$$

Since the column vectors (51) are solutions of the system (28), we have

$$(59) \quad \phi_{ij}^+(t) = \zeta_i(t) v_{mj}^+(t) - f^+(t) \overline{v_{ij}^+(t)} \in \mathcal{P}^+, \quad 1 \leq j \leq m, \quad 1 \leq i < m.$$

Direct computations give (see (57), (49), and (59))

$$(F^{-1})^*(t) V(t) = (\overline{f^+(t)})^{-1} \begin{pmatrix} -\overline{\phi_{11}^+(t)} & -\overline{\phi_{12}^+(t)} & \cdots & -\overline{\phi_{1m}^+(t)} \\ -\overline{\phi_{21}^+(t)} & -\overline{\phi_{22}^+(t)} & \cdots & -\overline{\phi_{2m}^+(t)} \\ \vdots & \vdots & \vdots & \vdots \\ -\overline{\phi_{m-1,1}^+(t)} & -\overline{\phi_{m-1,2}^+(t)} & \cdots & -\overline{\phi_{m-1,m}^+(t)} \\ \overline{v_{m1}^+(t)} & \overline{v_{m2}^+(t)} & \cdots & \overline{v_{mm}^+(t)} \end{pmatrix}.$$

Thus, there exists a matrix function  $\Phi^+(t) \in \mathcal{P}^+$  (hence

$$(60) \quad \det \Phi^+(z) \in \mathcal{P}^+)$$

such that

$$(F^{-1})^*(t) V(t) = (\overline{f^+(t)})^{-1} \cdot \overline{\Phi^+(t)}.$$

Consequently (see (58)),

$$(\overline{f^+(t)})^{-1} \det V(t) = (\overline{f^+(t)})^{-m} \det \overline{\Phi^+(t)}$$

so that

$$(f^+(t))^{m-1} \det \overline{V(t)} = \det \Phi^+(t).$$

Thus, it follows from (21), (55), and (60) that

$$(61) \quad \det \overline{V(t)} \in \mathcal{P}^+.$$

The relations (56) and (61) imply  $\det V(t) \in \mathcal{P}^+ \cap \mathcal{P}^-$  yielding (54).

The matrix function  $V(t)$  is not yet unitary, but it can be easily made such by multiplying from the right by a constant matrix. Namely, the matrix  $C = (c_{ij})_{i,j=1,2,\dots,m}$  defined by (53),

$$(62) \quad C = (V^*(t)V(t))^T, \quad t \in \mathbb{T},$$

is nonsingular. Indeed, if  $C$  were singular and  $\mathbf{0} \neq \mathbf{w} = (w_1, w_2, \dots, w_m) \in \mathbb{C}^m$  were such that  $\mathbf{w}C = 0$ , then

$$\left\| \sum_{j=1}^m w_j V^j(t) \right\|_{\mathbb{C}^m}^2 = \mathbf{w}C\mathbf{w}^* = 0$$

for each  $t \in \mathbb{T}$ , i.e. the vector functions  $V^1(t), V^2(t), \dots, V^m(t)$  would be linearly dependent. But this is impossible since the linear functional  $\mathbf{L} : L_\infty^+ \times L_\infty^+ \times \dots \times L_\infty^+ \rightarrow \mathbb{C}^m$  which maps  $(x_1^+(t), x_2^+(t), \dots, x_m^+(t))$  into the 0th Fourier coefficients of the functions standing on the left-hand side of the system (28), i.e. into  $(c_0\{\zeta_1(t)x_m^+(t) - f^+(t)\overline{x_1^+(t)}\}, \dots, c_0\{\zeta_{m-1}(t)x_m^+(t) - f^+(t)\overline{x_{m-1}^+(t)}\}, c_0\{\zeta_1(t)x_1^+(t) + \zeta_2(t)x_2^+(t) + \dots + f^+(t)\overline{x_m^+(t)}\})$ , transforms  $m$  vector functions  $V^1(t), V^2(t), \dots, V^m(t)$  into linearly independent standard bases of  $\mathbb{C}^m$ , namely,  $\mathbf{L}(V^j(t)) = (\delta_{j1}, \delta_{j2}, \dots, \delta_{jm})$ ,  $j = 1, 2, \dots, m$ , because of (38). Consequently,  $V(1)$  is also nonsingular since  $C^T = V^*(1)V(1)$  (see (62)). Let

$$(63) \quad U(t) = V(t)(V(1))^{-1}.$$

Then  $U(t)$  is unitary since (see (63), (62))

$$U^*(t)U(t) = ((V(1))^{-1})^* V^*(t)V(t)(V(1))^{-1} = ((V(1))^{-1})^* V^*(1)V(1)(V(1))^{-1} = I_m.$$

Since the matrix  $(V(1))^{-1}$  is constant,  $U(t) \in \mathcal{U}_m(\mathbb{T})$  has the same structure (15), (23) as  $V(t)$ , and

$$(64) \quad F(t)U(t) \in \mathcal{P}^+$$

holds as well (see (52) and (63)). Moreover,  $\det U(t) = \text{const}$ ,  $t \in \mathbb{T}$  (see (54) and (63)), which implies that  $\det U(t) = 1$  as we have  $U(1) = I_m$  (see (63)). Consequently,

$$(65) \quad U(t) \in \mathcal{SU}_m(\mathbb{T}).$$

Let now

$$(66) \quad U_F(t) = U(t) \cdot (FU(0))^{-1} \sqrt{FU(0)(FU(0))^*}.$$

The multiplier of  $U(t)$  in (66) is a (constant) unitary matrix, so that  $U_F(t) \in \mathcal{U}_m(\mathbb{T})$ , it has the structure (15), (23) (since  $U(t)$  has this structure), the inclusion (24) holds (see (64), (66)), and (25) is valid too (see (66)). The relation  $\det U_F(t) = 1$  holds since  $\det U_F(t) = c$  where  $|c| = 1$  (see (65), (66)), while  $c > 0$  since we know that  $0 < \det(FU_F)(0) = c \cdot \det F(0) = cf^+(0)$  (see (25), Corollary 1, and (18)) and

$f^+(0) > 0$  (see (21)). Consequently, the matrix function  $U_F(t) \in \mathcal{SU}_m(\mathbb{T})$  defined by (66) satisfies the requirements of Theorem 2 (a) and it has been constructed explicitly. The proof of the part (a) is finished.

REMARK 2. Note that, as in the case of  $U(t)$ , the modified column vectors of  $U_F(t)$  are solutions of the system (28) since this property of a matrix function is preserved when we multiply it by a constant matrix from the right.

## 6. DESCRIPTION OF THE METHOD

A brief outline of the method is the following:  $S(t)$  is approximated by  $M_N(t)M_N^*(t)$ , where  $M_N(t)$  is a lower triangular matrix function with analytic entries on the diagonal and whose entries below the diagonal have only finite number of nonzero Fourier coefficients with negative indices, the last product is represented as  $M_N^+(t)(M_N^+)^*(t)$ , where an analytic matrix function  $M_N^+(t)$  is constructed explicitly, and its convergence to  $S^+(t)$  is proved.

Given a matrix spectral density (1), first the lower-upper triangular factorization of  $S(t)$  is performed,

$$(67) \quad S(t) = M(t)(M(t))^*,$$

where

$$(68) \quad M(t) = \begin{pmatrix} f_1^+(t) & 0 & \cdots & 0 & 0 \\ \xi_{21}(t) & f_2^+(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{r-1,1}(t) & \xi_{r-1,2}(t) & \cdots & f_{r-1}^+(t) & 0 \\ \xi_{r1}(t) & \xi_{r2}(t) & \cdots & \xi_{r,r-1}(t) & f_r^+(t) \end{pmatrix}.$$

The functions  $f_m^+(t)$ ,  $m = 1, 2, \dots, r$ , on the diagonal are taken the canonical spectral factors of the positive functions  $\det S_m(t)/\det S_{m-1}(t)$ , where  $S_0(t) = 1$  and  $S_m(t) = (S(t))_{m \times m}$ , the upper-left  $m \times m$  submatrix of  $S(t)$ . Namely,

$$(69) \quad f_m^+(z) = \frac{(\det S_m)^+(z)}{(\det S_{m-1})^+(z)},$$

where (see (4))

$$(\det S_m)^+(z) = \exp \left( \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \det S_m(e^{i\theta}) d\theta \right).$$

We have  $\log \det S_m(t) \in L_1(\mathbb{T})$ ,  $m = 0, 1, \dots, r$ , by virtue of (2) (see, e.g., [4; Sect. 5]), so that the functions  $(\det S_m)^+(z)$ , and consequently  $f_m^+(z)$ ,  $m = 1, 2, \dots, r$ , are well defined in (69). The entries  $\xi_{ij}$ ,  $2 \leq i \leq r$ ,  $1 \leq j < i$ , can be found in a standard algebraic way from the relation (67).

Note that (67) implies  $|f_1^+|^2 = s_{11} \in L_1$  and  $\sum_{j=1}^{i-1} |\xi_{ij}|^2 + |f_i^+|^2 = s_{ii} \in L_1$ ,  $i = 2, 3, \dots, r$ . Thus  $M(t) \in L_2(\mathbb{T})$  (and hence  $M(t)U(t) \in L_2(\mathbb{T})$  for any  $U(t) \in \mathcal{U}_r(\mathbb{T})$ ). Furthermore,  $f_m^+ \in \mathcal{O}_2^O$ ,  $m = 1, 2, \dots, r$ , which implies that

$$(70) \quad \det M(t) = f_1^+(t)f_2^+(t) \cdots f_r^+(t) \in \mathcal{O}_{2/r}^O.$$

We search for  $U(t) \in \mathcal{SU}_r(\mathbb{T})$  such that  $M(t)U(t)$  is a spectral factor of  $S(t)$  and continue the description of our method in terms of Propositions 1 and 2, which can be easily proved using Theorems 1 and 2, respectively.

**Proposition 1.** *A spectral factor of  $S(t)$  can be represented as*

$$(71) \quad S^+(t) = M(t)\mathbf{U}_2(t)\mathbf{U}_3(t)\dots\mathbf{U}_r(t),$$

where  $\mathbf{U}_m(t) \in \mathcal{SU}_r(\mathbb{T})$  has the block matrix form

$$(72) \quad \mathbf{U}_m(t) = \begin{pmatrix} U_{F_m}(t) & 0 \\ 0 & I_{r-m} \end{pmatrix}, \quad m = 2, 3, \dots, r-1, \quad \mathbf{U}_r(t) = U_{F_r}(t),$$

$F_m$  in (72) is the matrix function of the form (13) whose last row coincides with the last row of  $(M_{m-1}(t))_{m \times m}$ ,

$$(73) \quad M_1(t) := M(t), \quad M_m(t) := M(t)\mathbf{U}_2(t)\mathbf{U}_3(t)\dots\mathbf{U}_m(t) = M_{m-1}(t)\mathbf{U}_m(t),$$

and  $U_{F_m}(t) \in \mathcal{SU}_m(\mathbb{T})$  is the corresponding matrix function determined according to Theorem 1,  $m = 2, 3, \dots, r$ .

*Proof.* Obviously, the product of two matrix functions from  $\mathcal{SU}_r(\mathbb{T})$  is in the same class. Thus, by virtue of Lemma 2 (see (67), (70)), it suffices to show that

$$(74) \quad S^+(t) = M_r(t) = M(t)\mathbf{U}_2(t)\mathbf{U}_3(t)\dots\mathbf{U}_r(t) \in L_2^+(\mathbb{T}).$$

It follows from the structures of the matrices in (72) and (73) that

$$(75) \quad (M_m(t))_{m \times m} = (M_{m-1}(t))_{m \times m} U_{F_m}(t),$$

while the last  $r - m$  columns of  $M(t)$  remains unaltered in  $M_m(t)$ .

We show by induction that

$$(76) \quad (M_m(t))_{m \times m} \in L_2^+(\mathbb{T}), \quad m = 1, 2, \dots, r.$$

Indeed, clearly (76) is correct for  $m = 1$ . Assume now that (76) holds when  $m$  is replaced by  $m - 1$  in it, i.e.

$$(77) \quad (M_{m-1}(t))_{(m-1) \times (m-1)} \in L_2^+(\mathbb{T}).$$

Then  $(M_{m-1}(t))_{m \times m} \in L_2(\mathbb{T})$  has the form

$$(78) \quad (M_{m-1}(t))_{m \times m} = \begin{pmatrix} \mu_{11}^+(t) & \mu_{12}^+(t) & \cdots & \mu_{1,m-1}^+(t) & 0 \\ \mu_{21}^+(t) & \mu_{22}^+(t) & \cdots & \mu_{2,m-1}^+(t) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{m-1,1}^+(t) & \mu_{m-1,2}^+(t) & \cdots & \mu_{m-1,m-1}^+(t) & 0 \\ \zeta_1(t) & \zeta_2(t) & \cdots & \zeta_{m-1}(t) & f_m^+(t) \end{pmatrix},$$

where  $\mu_{ij}^+(t) \in L_2^+(\mathbb{T})$ ,  $i, j = 1, 2, \dots, m-1$ , by (77),  $\zeta_j(t) \in L_2(\mathbb{T})$ ,  $j = 1, 2, \dots, m-1$ ,  $f_m^+ \in \mathcal{O}_2^0$  is defined by (69) (see (68)), and  $(\zeta_1(t), \zeta_2(t), \dots, \zeta_{m-1}(t), f_m^+(t))$  is the last row of  $F_m(t)$  according to its definition in Proposition 1 (note that Theorem 1 can

be applied to the matrix function  $F_m(t)$ ). The direct computation shows that (78) is equal to (see (13))

$$\begin{pmatrix} \mu_{11}^+(t) & \cdots & \mu_{1,m-1}^+(t) & 0 \\ \mu_{21}^+(t) & \cdots & \mu_{2,m-1}^+(t) & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{m-1,1}^+(t) & \cdots & \mu_{m-1,m-1}^+(t) & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} F_m(t) = \begin{pmatrix} (M_{m-1}(t))_{(m-1) \times (m-1)} & 0 \\ 0 & 1 \end{pmatrix} F_m(t)$$

and we get that (see (75))  
(79)

$$(M_m(t))_{m \times m} = (M_{m-1}(t))_{m \times m} U_{F_m}(t) = \begin{pmatrix} (M_{m-1}(t))_{(m-1) \times (m-1)} & 0 \\ 0 & 1 \end{pmatrix} F_m(t) U_{F_m}(t)$$

belongs to  $L_2^+(\mathbb{T})$  (see (77) and (17)). Thus (76) is valid and taking  $m = r$  in (76) we get (74).

Proposition 1 is proved.  $\square$

REMARK 3. One can see from the above proof that  $(M_m(t))_{m \times m}$  is a spectral factor of  $S_m(t)$ ,

$$S_m^+(t) = (M_m(t))_{m \times m}, \quad m = 1, 2, \dots, r.$$

Thus the representation (71) realizes the step-by-step factorization of the upper-left submatrices of  $S(t)$ .

REMARK 4. There is an alternative way of representation (71) which avoids preliminary computation of entries  $\xi_{ij}(t)$  in (68). Namely, we can compute only  $f_m^+(t)$ ,  $m = 1, 2, \dots, r$ , in (68) according to (69) (note that  $(M_1(t))_{1 \times 1} = f_1^+(t)$ ) and determine  $(M_m(t))_{m \times m}$  recurrently from  $(M_{m-1}(t))_{(m-1) \times (m-1)}$  by the formula (79). The entries  $\zeta_1(t), \zeta_2(t), \dots, \zeta_{m-1}(t)$  of  $F_m(t)$  and of (78) can be determined from the equation

$$(M_{m-1}(t))_{(m-1) \times (m-1)} (\zeta_1, \zeta_2, \dots, \zeta_{m-1})^* = (s_{1m}, s_{2m}, \dots, s_{m-1,m})^T$$

which follows from  $(M_{m-1}(t))_{m \times m} (M_{m-1}(t))_{m \times m}^* = S_m(t)$ . In this way, we can obtain each  $(M_m(t))_{m \times m}$ ,  $m = 1, 2, \dots, r$ , and respectively  $S^+(t) = (M_r(t))_{r \times r} = M_r(t)$ .

Relying on Proposition 1, we recurrently approximate  $S^+(t)$  as follows. Let  $N_2, N_3, \dots, N_r$  be large positive integers, and let

$$(80) \quad \hat{S}^+(t) = \hat{S}^+[N_2, N_3, \dots, N_r] := M(t) \hat{\mathbf{U}}_2(t) \hat{\mathbf{U}}_3(t) \dots \hat{\mathbf{U}}_r(t),$$

where  $\hat{\mathbf{U}}_m(t) \in \mathcal{SU}_r(\mathbb{T})$  has the block matrix form

$$(81) \quad \hat{\mathbf{U}}_m(t) = \begin{pmatrix} U_{\hat{F}_m^{(N_m)}}(t) & 0 \\ 0 & I_{r-m} \end{pmatrix}, \quad m = 2, 3, \dots, r-1, \quad \hat{\mathbf{U}}_r(t) = U_{\hat{F}_r^{(N_r)}}(t),$$

$\hat{F}_m \in L_2(\mathbb{T})$  is the matrix function of the form (13) whose last row coincides with the last row of  $(\hat{M}_{m-1}(t))_{m \times m}$ ,

$$(82) \quad \hat{M}_1(t) := M(t), \quad \hat{M}_m(t) := M(t) \hat{\mathbf{U}}_2(t) \hat{\mathbf{U}}_3(t) \dots \hat{\mathbf{U}}_m(t) = \hat{M}_{m-1}(t) \hat{\mathbf{U}}_m(t),$$



$\hat{F}_m^{(N_m)}(t)$  in (81) is  $P_{N_m}\hat{F}_m(t)$  (see the definition of the projection operator  $P_N$  in Section 3), and  $U_{\hat{F}_m^{(N_m)}}(t) \in \mathcal{SU}_m(\mathbb{T})$  is the corresponding matrix function determined according to Theorem 2 (a),  $m = 2, 3, \dots, r$ . We emphasize that as  $S(t)$  is given and the positive integers  $N_2, N_3, \dots, N_r$  are fixed, each  $\hat{M}_m(t) = \hat{M}_m[N_2, N_3, \dots, N_m]$ ,  $m = 2, 3, \dots, r$ , can be explicitly constructed according to the proof of Theorem 2 (a). The following proposition shows that

$$(83) \quad \hat{S}^+(t) = \hat{S}^+[N_2, N_3, \dots, N_r] = \hat{M}_r(t)$$

(see (80) and (82)) approximates a spectral factor of  $S(t)$ .

**Proposition 2.**  $\|S^+(t) - \hat{S}^+(t)\|_{L_2} \rightarrow 0$  as  $N_2, N_3, \dots, N_r \rightarrow \infty$ .

*Proof.* We prove by induction that

$$(84) \quad \|M_m(t) - \hat{M}_m(t)\|_{L_2} \rightarrow 0 \text{ as } N_2, N_3, \dots, N_m \rightarrow \infty, \quad m = 2, 3, \dots, r.$$

Indeed, assume that

$$(85) \quad \|M_{m-1}(t) - \hat{M}_{m-1}(t)\|_{L_2} \rightarrow 0 \text{ as } N_2, N_3, \dots, N_{m-1} \rightarrow \infty$$

holds (note that  $M_1(t) = \hat{M}_1(t)$ ). Then, by virtue of the definitions of  $F_m(t)$  and  $\hat{F}_m(t)$ ,

$$(86) \quad \|F_m(t) - \hat{F}_m(t)\|_{L_2} \rightarrow 0 \text{ as } N_2, N_3, \dots, N_{m-1} \rightarrow \infty.$$

Obviously (see (20)),

$$(87) \quad \|\hat{F}_m(t) - \hat{F}_m^{(N)}(t)\|_{L_2} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

It follows from (86) and (87) that

$$\|F_m(t) - \hat{F}_m^{(N_m)}(t)\|_{L_2} \rightarrow 0 \text{ as } N_2, N_3, \dots, N_m \rightarrow \infty.$$

Thus, by virtue of Theorem 2 (b),  $U_{\hat{F}_m^{(N_m)}}(t) \rightrightarrows U_{F_m}(t)$  and hence (see (72) and (81))  $\hat{\mathbf{U}}_m(t) \rightrightarrows \mathbf{U}_m(t)$  as  $N_2, N_3, \dots, N_m \rightarrow \infty$ . Consequently (see (73), (82), (85), and Statement 1 in Sect 3),

$$\|M_m(t) - \hat{M}_m(t)\|_{L_2} = \|M_{m-1}(t)\mathbf{U}_m(t) - \hat{M}_{m-1}(t)\hat{\mathbf{U}}_m(t)\|_{L_2} \rightarrow 0$$

as  $N_2, N_3, \dots, N_m \rightarrow \infty$  and (84) holds.

If we substitute  $m = r$  into (84), we get the proposition (see (74) and (83)).  $\square$

**REMARK 5.** *The rate of convergence in Proposition 2 is estimated under minor restrictions on  $S(t)$ , which is the subject of a forthcoming paper.*

**REMARK 6.** *In actual computations of  $\hat{S}^+(t)$  according to (80), we cannot take  $M(t)$  exactly since it requires scalar spectral factorizations. As it was mentioned above, our method does not contain any improvement in approximate computation of  $M(t)$ . We can assume that it can be constructed  $\hat{M}_1(t) = \hat{M}_1[N_1](t)$  in (82) such that  $\|M(t) - \hat{M}_1(t)\|_{L_2} \rightarrow 0$  as  $N_1 \rightarrow \infty$ , and the rest of the proof of Proposition 2 goes through without any change.*

If we wish to construct an approximation to the canonical spectral factor  $S_c^+$ , then we take (see Lemma 1)  $\hat{S}_c^+(z) = \hat{S}^+(z)(\hat{S}^+(0))^{-1}\sqrt{\hat{S}^+(0)(\hat{S}^+(0))^*}$ .

## 7. NUMERICAL SIMULATIONS

The computer code for the factorization of polynomial matrix functions by our method was written in MatLab in order to test the algorithm numerically and compare it with other existing software implementations available in the MatLab toolbox “Polyx”. The results of our numerical simulations are presented in this section.

Two different commands,  $\text{spf}(\cdot)$  and  $\text{spf}(\cdot, \text{syl})$  are available in Polyx to perform polynomial matrix spectral factorization for a discrete time variable  $z$ . (As it is explained in the software manual these factorizations are based on the Newton-Raphson iteration and on the Sylvester’s method, respectively.) We have supplied the three programs with the same data and compared their performances. The computer with characteristics Intel(R) Core(TM) Quad CPU, Q6600 2.40GHz, 2.40 GHz, RAM 2.00Gb was used for these simulations.

In the first place we took a test matrix whose spectral factorization was known beforehand,

$$\begin{pmatrix} 2z^{-1} + 6 + 2z & 7z^{-1} + 22 + 11z \\ 11z^{-1} + 22 + 7z & 38z^{-1} + 84 + 38z \end{pmatrix} = \begin{pmatrix} 2 + z^{-1} & 1 \\ 7 + 5z^{-1} & 3 + z^{-1} \end{pmatrix} \begin{pmatrix} 2 + z & 7 + 5z \\ 1 & 3 + z \end{pmatrix}$$

(the matrix is very simple, but its determinant,  $-z^{-2} + 2 - z^2$ , has two double zeros on the boundary, which usually causes difficulties in many methods). So the correct answer for the (right) spectral factor (with the uniqueness restriction for the coefficient matrix of the highest degree of  $z$  to be upper triangular with positive entries on the diagonal, as it is in Polyx) is

$$\begin{pmatrix} 2.000\dots & 7.000\dots \\ 1.000\dots & 3.000\dots \end{pmatrix} + \begin{pmatrix} 1.000\dots & 5.000\dots \\ 0 & 1.000\dots \end{pmatrix} t.$$

The resulting coefficient matrices obtained by  $\text{spf}(\cdot)$  and  $\text{spf}(\cdot, \text{syl})$  were the same

$$\begin{pmatrix} 1.998657938438840 & 6.995302784535943 \\ 1.002013092341746 & 3.007856444481516 \end{pmatrix}, \begin{pmatrix} 1.000671030780579 & 5.001537986446627 \\ 0 & 1.001621242570816 \end{pmatrix}$$

and the time elapsed varied within 0.22-0.24 sec. Below we present the results of computation by the program based on our algorithm which shows the advantage of the proposed method. In the process of the calculations three different pairs of tuning parameters were used:  $\epsilon$ , the accuracy level of scalar spectral factorizations of  $S_1(t)$  and  $\det S_2(t)$  in (69), and  $N = N_2$ , a positive integer in (83). Accuracy improvements are evident as proved theoretically in Section 6:

$\epsilon = 0.0001$ ;  $N = 20$ ; time elapsed: 0.04 sec.

$$\begin{pmatrix} 1.999540036680776 & 6.998506647352555 \\ 1.000742648478974 & 3.002561886914702 \end{pmatrix}, \begin{pmatrix} 1.000213381614498 & 5.000347044583584 \\ 0 & 1.000664442251186 \end{pmatrix}$$

$\epsilon = 0.000001$ ;  $N = 30$ ; time elapsed: 0.14 sec

$$\begin{pmatrix} 1.999999670218186 & 6.999998845763626 \\ 1.000000494672526 & 3.000001951212981 \end{pmatrix}, \begin{pmatrix} 1.000000439718603 & 5.000000357258912 \\ 0 & 1.000000164890947 \end{pmatrix}$$

$\epsilon = 0.00000001$ ;  $N = 40$ ; time elapsed: 0.31 sec

$$\begin{pmatrix} 1.999999957974826 & 6.999999852911864 \\ 1.000000063037774 & 3.000000246615687 \end{pmatrix}, \begin{pmatrix} 1.000000021012597 & 5.000000047560575 \\ 0 & 1.000000051966992 \end{pmatrix}$$

When data were selected at random and exact results were unknown, the mean of absolute values of polynomial coefficients of the error matrix  $(\hat{S}^+)^* \hat{S}^+ - S$  was taken in the capacity of an accuracy estimator (in general, the closeness of  $(\hat{S}^+)^* \hat{S}^+$  to  $S$  does not imply that  $\hat{S}^+$  is close to  $S^+$ , see [13], [1], but this is the case for polynomial matrix functions). In the table below this mean is denoted by  $\varepsilon$ . Calculation time values are shown, and the matrix sizes are given; say  $4 \times 10$  indicates that a  $4 \times 4$  test matrix was selected with (Laurent) polynomial entries of degree 10 (with coefficients from -10 to 10). The results of calculations by  $\text{spf}(\cdot)$  and  $\text{spf}(\cdot, \text{syl})$  were almost identical. In the case of our algorithm, we varied the tuning parameters of the program ( $N_2, N_3, \dots, N_r$  in (83)) so as to obtain a slightly higher accuracy than by  $\text{spf}(\cdot)$  and  $\text{spf}(\cdot, \text{syl})$ , while the advantage in time was noticeable.

	matr. size	time sec	accur. $\varepsilon$	matr. size	time sec	accur. $\varepsilon$	matr. size	time sec	accur. $\varepsilon$	matr. size	time sec	accur. $\varepsilon$
$\text{spf}(\cdot)$	4x10	0.67	$10^{-10}$	6x15	5.8	$10^{-6}$	10x20	218	$10^{-6}$	15x20	1949	$10^{-7}$
$\text{spf}(\cdot, \text{syl})$	—	0.56	$10^{-10}$	—	4.7	$10^{-6}$	—	214	$10^{-6}$	—	1952	$10^{-7}$
New Alg.	—	0.46	$10^{-12}$	—	3.4	$10^{-7}$	—	65	$10^{-8}$	—	216	$10^{-8}$

We express our gratitude to PhD student Vakhtang Rodonaia for working out the software for testing our algorithm and collecting the numerical data.

## 8. CONCLUSION

A new algorithm of matrix spectral factorization is developed, which factorizes any matrix spectral density that admits spectral factorization. The advantage of the algorithm is illustrated by the examples of numerical simulations.

## 9. APPENDICES

**A. Convergence properties.** In this appendix we continue the proof of Theorem 2 started in Sect. 5 and prove the second part (b), which deals with convergence properties of the algorithm. This proof is similar to the one given in [12] for the two-dimensional case.

Observe first that:

(i) *if  $\{U_{F(N)}(t)\}_{N \in \mathbb{N}_0}$ ,  $\mathbb{N}_0 \subset \mathbb{N}$ , is any convergent almost everywhere subsequence of  $U_{F(N)}(t)$ , i.e. if*

$$U_{F(N)}(t) \rightarrow U(t) \text{ a.e. as } \mathbb{N}_0 \ni N \rightarrow \infty,$$

*then*

$$(88) \quad F_c^+(t) = F(t)U(t).$$

Indeed, passing to the limit in the relations (22), (24),  $\det(F^{(N)}U_{F(N)})(z) = f_{(N)}^+(z)$  (see Corollary 1 and (18)), and (25), we get

$$U(t) \in \mathcal{SU}_m(\mathbb{T}), \quad F(t)U(t) \in L_2^+(\mathbb{T}), \quad (FU)(z) = f^+(z) \in \mathcal{O}_2^0 \text{ (see (14)), and } FU(0) > 0,$$

which implies (88) (see Lemma 2).

Now it will be shown that

(ii) from each subsequence  $\{U_{F(N)}(t)\}_{N \in \mathbb{N}_1 \subset \mathbb{N}}$  we can extract an a.e. convergent subsequence  $\{U_{F(N)}(t)\}_{N \in \mathbb{N}_0 \subset \mathbb{N}_1}$ .

This will finish the proof of the relation (26) by virtue of the uniqueness of the canonical spectral factor and the property (i).

We say that a sequence of functions  $f_n \in L_2$ ,  $n = 1, 2, \dots$  belongs to  $\mathcal{K}$ ,  $\{f_n\}_{n \in \mathbb{N}} \in \mathcal{K}$ , if one can extract a convergent in  $L_2$  subsequence from  $f_n$ . Recall that an operator  $K : L_\infty^\pm \rightarrow L_2^\pm$  is called compact if  $\{K(h_n)\}_{n \in \mathbb{N}} \in \mathcal{K}$  for any bounded sequence  $\{h_n\}_{n \in \mathbb{N}}$ ,  $|h_n| < c$ ,  $n = 1, 2, \dots$  (see [15; §4.6]).

To prove the property (ii), observe that Hankel's operators

$$H_\zeta : L_\infty^+ \rightarrow L_2^-, \quad \zeta \in L_2, \quad \text{and} \quad H_f^* : L_\infty^- \rightarrow L_2^+, \quad f \in L_2,$$

defined by

$$(89) \quad H_\zeta(u^+) = P^-(\zeta u^+), \quad u^+ \in L_\infty^+,$$

and

$$(90) \quad H_f^*(u^-) = P^+(f u^-), \quad u^- \in L_\infty^-,$$

are compact operators as limits of finite-dimensional operators (see, e.g., [15; Th. 4.6.1]).

Fix arbitrary  $j \leq m$ , and let  $(u_1^{+(N)}, u_2^{+(N)}, \dots, u_{m-1}^{+(N)}, \overline{u_m^{+(N)}})^T$  be the  $j$ th column of  $U_{F(N)}$ . Since the modified columns of  $U_{F(N)}(t)$  are solutions of the system (28) (see Remark 2 in Sect. 5), we have

$$(91) \quad \zeta_i^{(N)} u_m^{+(N)} - f_{(N)}^+ \overline{u_i^{+(N)}} \in \mathcal{P}^+, \quad 1 \leq i \leq m-1,$$

and

$$(92) \quad \zeta_1^{(N)} u_1^{+(N)} + \zeta_2^{(N)} u_2^{+(N)} + \dots + \zeta_{m-1}^{(N)} u_{m-1}^{+(N)} + f_{(N)}^+ \overline{u_m^{+(N)}} \in \mathcal{P}^+.$$

It follows from the compactness of the operator (89) and (19) that

$$(93) \quad \left\{ P^-(\zeta_i^{(N)} u_i^{+(N)}) \right\}_{N \in \mathbb{N}_1} = \left\{ P^-(\zeta_i^{(N)} - \zeta_i) u_i^{+(N)} + P^-(\zeta_i u_i^{+(N)}) \right\}_{N \in \mathbb{N}_1} \in \mathcal{K}$$

for each  $i = 1, 2, \dots, m-1$ , and thus  $\{P^-(f_{(N)}^+ \overline{u_m^{+(N)}})\}_{N \in \mathbb{N}_1} \in \mathcal{K}$ , because of the relation (92). It follows from the compactness of operator (90) and (19) that

$$(94) \quad \left\{ P^+(f_{(N)}^+ \overline{u_m^{+(N)}}) \right\}_{N \in \mathbb{N}_1} = \left\{ P^+((f_{(N)}^+ - f^+) \overline{u_m^{+(N)}}) + P^+(f^+ \overline{u_m^{+(N)}}) \right\}_{N \in \mathbb{N}_1} \in \mathcal{K}$$

as well. Hence (see (93), (94))

$$(95) \quad \left\{ f_{(N)}^+ \overline{u_m^{+(N)}} \right\}_{N \in \mathbb{N}_1} = \left\{ P^+(f_{(N)}^+ \overline{u_m^{+(N)}}) + P^-(f_{(N)}^+ \overline{u_m^{+(N)}}) - c_0(f_{(N)}^+ \overline{u_m^{+(N)}}) \right\}_{N \in \mathbb{N}_1} \in \mathcal{K}.$$

Since  $f_{(N)}^+(t) \rightrightarrows f^+(t)$  and  $f^+(t) \neq 0$  for a.a.  $t \in \mathbb{T}$  (see (14)), it follows from (95) that  $\{u_m^{+(N)}\}_{N \in \mathbb{N}_1}$  contains an almost everywhere convergent subsequence.

Now we will show that the same is true for  $\{u_i^{+(N)}\}_{N \in \mathbb{N}_1}$ ,  $1 \leq i \leq m-1$ . Since  $\{\zeta_i^{(N)} u_m^{+(N)}\}_{N \in \mathbb{N}_1} \in \mathcal{K}$  (see (19) and Statement 1) and hence  $\{P^-(\zeta_i^{(N)} u_m^{+(N)})\}_{N \in \mathbb{N}_1} \in \mathcal{K}$ , it follows from (91) that

$$(96) \quad \left\{ P^-(f_{(N)}^+ \overline{u_i^{+(N)}}) \right\}_{N \in \mathbb{N}_1} \in \mathcal{K}$$

as well. The compactness of the operator (90) and (19) imply that

$$(97) \quad \left\{ P^+(f_{(N)}^+ \overline{u_i^{+(N)}}) \right\}_{N \in \mathbb{N}_1} = \left\{ P^+((f_{(N)}^+ - f^+) \overline{u_i^{+(N)}}) + P^+(f^+ \overline{u_i^{+(N)}}) \right\}_{N \in \mathbb{N}_1} \in \mathcal{K}$$

The relations (96) and (97) imply that  $\{f_{(N)}^+ \overline{u_i^{+(N)}}\}_{N \in \mathbb{N}_1} \in \mathcal{K}$  and, consequently, an almost everywhere convergent subsequence can be extracted from  $\{\overline{u_i^{+(N)}}\}_{N \in \mathbb{N}_1}$ .

The proof of the property (ii) is completed and thus (26) holds.

The proof of the remaining conditions in the part (b) continues as follows. Since the inverse of a matrix function  $F$  of the form (13) is

$$(98) \quad F^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ -\frac{\zeta_1}{f^+} & -\frac{\zeta_2}{f^+} & \cdots & -\frac{\zeta_{m-1}}{f^+} & \frac{1}{f^+} \end{pmatrix},$$

(19) implies that

$$(99) \quad (F^{(N)})^{-1}(t) \rightrightarrows F^{-1}(t).$$

Hence

$$U_{F^{(N)}}(t) = (F^{(N)})^{-1}(t) \cdot F^{(N)}(t) \cdot U_{F^{(N)}}(t) \rightrightarrows F^{-1}(t) F_c^+(t)$$

(see (99) and (26)) and if we denote  $U_F(t) := F^{-1}(t) F_c^+(t)$ , then (27) holds and the equation in (17) follows directly. Since each matrix function in (22) has the structure (15), (23), the limiting matrix function  $U_F(t) \in \mathcal{SU}_m(\mathbb{T})$  has the structure (15), (16). The uniqueness of  $U_F(t)$  follows from the uniqueness of the canonical spectral factor and the equation in (17) since  $F(t)$  is invertible.

**B. Displacement Structure.** In this section we prove that the matrix  $\Delta$  defined by (47) has a *displacement structure of rank  $m$  with respect to  $Z$* , i.e. (see [14; App. F.1])

$$(100) \quad R_Z \Delta := \Delta - Z \Delta Z^*$$

has rank  $m$ , where  $Z$  is the upper triangular  $(N+1) \times (N+1)$  matrix with ones on the first up-diagonal and zeros elsewhere (i.e. a Jordan block with eigenvalue 0). There are several forms of displacement structure and we have selected a suitable one.

Obviously,  $I_{N+1}$  has the displacement structure of rank 1, namely,

$$(101) \quad R_Z I_{N+1} = I_{N+1} - Z I_{N+1} Z^* = \mathcal{E} \mathcal{E}^*,$$

where  $\mathcal{E} = (0, 0, \dots, 0, 1)^T \in \mathbb{C}^{N+1}$ . We will show that for each Toeplitz-like matrix

$$(102) \quad \Theta = \begin{pmatrix} \eta_0 & \eta_1 & \eta_2 & \cdots & \eta_{m-1} & \eta_m \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_m & 0 \\ \eta_2 & \eta_3 & \eta_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \eta_m & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

the matrix  $\Theta\Theta^*$  has the displacement structure of rank 1, namely

$$(103) \quad R_Z(\Theta\Theta^*) = \Theta\Theta^* - Z\Theta\Theta^*Z^* = \Lambda\Lambda^*,$$

where  $\Lambda = (\eta_0, \eta_1, \dots, \eta_m)^T$ . Indeed, it follows from the definitions of matrices  $Z$ ,  $\mathcal{E}$ , and  $\Lambda$  and from the structure of  $\Theta$  that

$$(104) \quad Z^*\mathcal{E} = \mathbf{0} \quad \text{and} \quad \mathcal{E}^T Z = \mathbf{0}^T,$$

$$(105) \quad Z\Theta = \Theta Z^* \quad \text{and} \quad Z\Theta^* = \Theta^* Z^*,$$

and

$$(106) \quad \Theta - Z\Theta Z = \Lambda\mathbf{1}^T \quad \text{and} \quad \Theta^* - Z^*\Theta^*Z^* = \mathbf{1}\Lambda^*,$$

where  $\mathbf{0}$  and  $\mathbf{1}$  are defined by (40). Since  $\mathbf{1}^T \mathbf{1} = 1$ , it follows from (106) that

$$(\Theta - Z\Theta Z)(\Theta^* - Z^*\Theta^*Z^*) = \Lambda\Lambda^*.$$

Hence, taking into account (105),

$$\begin{aligned} \Lambda\Lambda^* &= \Theta\Theta^* - \Theta Z^*\Theta^*Z^* - Z\Theta Z\Theta^* + Z\Theta Z Z^*\Theta^*Z^* = \\ &= \Theta\Theta^* - Z\Theta\Theta^*Z^* - Z\Theta\Theta^*Z^* + Z\Theta Z Z^*\Theta^*Z^* \end{aligned}$$

and (103) holds since (see (101), (105), and (104))

$$\begin{aligned} -Z\Theta\Theta^*Z^* + Z\Theta Z Z^*\Theta^*Z^* &= Z\Theta(ZZ^* - I_{N+1})\Theta^*Z^* \\ &= -Z\Theta\mathcal{E}\mathcal{E}^T\Theta^*Z^* = -\Theta Z^*\mathcal{E}\mathcal{E}^T Z\Theta^* = -\Theta\mathbf{0}\mathbf{0}^T\Theta^* = 0. \end{aligned}$$

Every matrix  $\Theta_i$ ,  $i = 1, 2, \dots, m-1$ , defined by (45) has the structure (102) by virtue of (46). Thus we can write (103) for each  $i$ ,

$$(107) \quad R_Z(\Theta_i\Theta_i^*) = \Theta_i\Theta_i^* - Z\Theta_i\Theta_i^*Z^* = \Lambda_i\Lambda_i^*, \quad i = 1, 2, \dots, m-1.$$

Since  $R_Z$  defined by (100) is linear,  $R_Z(\Delta_1 + \Delta_2) = R_Z\Delta_1 + R_Z\Delta_2$ , it follows from (47), (107), and (101) that

$$R_Z\Delta = R_Z\left(\sum_{i=1}^{m-1} \Theta_i\Theta_i^* + I_{N+1}\right) = \sum_{i=1}^{m-1} R_Z(\Theta_i\Theta_i^*) + R_Z I_{N+1} = \sum_{i=1}^{m-1} \Lambda_i\Lambda_i^* + \mathcal{E}\mathcal{E}^* = A A^*,$$

where  $A = [\Lambda_1, \Lambda_2, \dots, \Lambda_{m-1}, \mathcal{E}]$  is the  $(N+1) \times m$  matrix (of rank at most  $m$ ) with columns  $\Lambda_1, \Lambda_2, \dots, \Lambda_{m-1}$  and  $\mathcal{E}$ .

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